

# Dynamics and Thermodynamics of (2+1)-Dimensional Evolving Lorentzian Wormhole

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**Abstract:** In this paper we study the relationship between the Einstein field equations for the (2+1)-dimensional evolving wormhole and the first law of thermodynamics. It has been shown that the Einstein field equations can be rewritten as a similar form of the first law of thermodynamics at the dynamical trapping horizon (as proposed by Hayward) for the dynamical spacetime which describes intrinsic thermal properties associated with the trapping horizon. For a particular choice of the shape and potential functions we are able to express field equations as a similar form of first law of thermodynamics  $dE = -TdS + WdA$  at the trapping horizons. Here  $E = \rho A$ ,  $T = -\kappa/2\pi$ ,  $S = 4\pi\tilde{r}_A$ ,  $W = (\rho - p)/2$ , and  $A = \pi\tilde{r}_A^2$ , are the total matter energy, horizon temperature, wormhole entropy, work density and volume of the evolving wormhole respectively.

**Keywords:** Wormhole; Thermodynamics; Entropy; Horizons.

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## Introduction

Black holes and wormholes solutions to the Einstein field equations have been studied to investigate the various physical properties of these objects [1]. In the past two decades, enormous work has been done in the revival of classical wormhole solutions. Historically, the idea of a wormhole was suggested by Flamm [2] by means of the standard embedding diagram. Later on a similar construction was attempted by Einstein and Rosen, so-called the Einstein-Rosen bridge [3]. Subsequently, it was shown that the bridge, also called a Schwarzschild wormhole is really a black hole [4]. But these wormholes were not traversable. Interest in the wormholes, which were both traversable and stable was stimulated by the seminal work of Morris and Thorne [5]. A static and spherically symmetric wormhole possesses intriguing geometrical structure having a throat that flares out in two opposite directions. The throat connects two arbitrary regions of the same spacetime or two distinct spacetimes. There are several reasons that arouse interest in these peculiar geometries. One of them is the possibility that these spacetimes could be thought of as time machines [6] which violate Hawking's chronology protection conjecture. Since the throat has the tendency to get closed in a very short time (of the order of Planck time), thereby restricting the time travel possibility. Another reason supports the possibility of motion of a particle through the throat. Since topologically, wormholes spacetimes are the same as that of black holes, but a minimal surface called the throat of wormhole is maintained in the time evolution, so that a traveler could pass through it in both directions. This could mainly be possible in the presence of an exotic matter (a matter which violates the null energy condition (NEC)) which holds up the wormhole structure and keeps the wormhole throat open.

In the light of above discussion, it is inevitable to seek a realistic matter which supports these exotic spacetimes. Recently, the astrophysical observations suggested that the cosmological fluid violates the NEC [7]. In the context of cosmology, the phantom energy with the equation of state  $\omega < -1$  is the most obvious choice that violates the NEC and is expected to be the source that could sustain the wormholes [8]. However, it has been argued that a wormhole could be constructed by letting the accretion of phantom energy onto black holes [9]. On the other hand, a wormhole may be converted to a black hole as the exotic matter gets evaporated from the wormhole's throat. So with this discussion, we come to the point that the wormholes are not merely the mathematical toy spacetime models with interest

only for science fiction movies, but also play as a model of plausible physical realities that might exist in the very spacetime fabric of our real universe. The understanding of this reality induced a revival in the study of wormhole spacetimes, particularly the consideration of accretion of phantom energy can significantly widen the radius of the throat [10]. Further, it has been argued that a wormhole could lead to an inflationary universe by absorbing large amount of exotic matter [11].

Most of the efforts regarding wormhole spacetimes have been devoted to study static configurations that must satisfy some specific properties in order to be traversable. However, one can study the wormhole configurations that are time dependent, such as rotating wormholes [12] or wormholes in a cosmological set up have been discussed in detail in [13–15].

In this paper, our prime aim is to discuss the thermodynamical feature of Einstein’s field equations for the (2+1)-dimensional evolving wormhole spacetime. Since in view of global properties, wormholes are quite distinct from the black holes. But according to Hayward, if one considers the local properties, both (black holes and wormholes) can be characterized by the presence of marginal (marginally trapped) surfaces, and certainly may be defined in terms of trapping horizons, which are nothing but hypersurfaces foliated by marginal surfaces [16–18]. So under this consideration (by defining trapping horizon), he constructed a formalism to explain the thermodynamical properties of spherical, dynamical black holes [19]. In the case of wormholes, the idea to study the thermodynamics can be motivated by the fact that even though the definition of an event horizon is no longer possible for the evolving wormhole, but we can still introduce the concept of a trapping horizon for these objects. Hence introducing the trapping horizon for wormhole spacetimes allows one to study these peculiar geometries unifying them with black holes [9]. Therefore, the idea that wormholes may show some characteristics and properties which are parallel to those already found in black holes, seems to be quite natural, including in particular ”wormhole thermodynamics” [9]. Recently, the authors [20] have shown that one can construct three laws of thermodynamics for Lorentzian wormholes by using trapping horizons. They demonstrated that these laws are related with a thermal phantom-like radiation process coming from the wormhole throat.

In literature, the work on wormhole spacetimes in lower [21] and higher dimensions have been studied by many authors. For instance, the Euclidean wormholes have been considered by Gonzales-Diaz and by Jianjun and Sicong [22]. The discussion on Lorentzian worm-

holes in the  $n$ -dimensional Einstein gravity or Einstein-Gauss-Bonnet theory of gravitation is available in [23] while the author [24] discussed the interesting features of static wormhole in higher dimensional cosmology by considering the scale factor  $a(t)$ . Mostly the authors construct the wormholes solutions in (3+1)-dimensional gravity. However, there are few solutions of wormhole in (2+1) dimensional gravity which is a covariant theory of gravity that has a great simplicity when compared with general relativity. This theory has been applied to study some quantum aspects of gravity [25]. As in the case of (2+1)-dimensional Banados, Teitelboim and Zanelli (BTZ) black hole [26], several authors have shown interest in wormholes in (2+1) dimensional gravity. Delgaty et al. [27] studied the characteristics of traversable wormholes in the (2+1) dimensional gravity with cosmological constant. Aminneborg et al. [28] compared the properties of wormhole with the characteristics of black hole while Kim et al. [29] gave two specific solutions of wormhole taking (2+1) dimensional gravity with a dilatonic field. Further Rahaman et al. [30] and Jamil and Farooq [31] constructed the phantom wormholes in the lower dimensional (2+1) gravity. In a similar fashion the thermodynamics of lower dimensional gravity has been studied to generalize the deep relation between gravity theories and thermodynamics [32]. To study this relation in a wide range of spacetime geometries, it is inevitable to extend it to the (2+1)-dimensional wormhole.

This paper is structured as follows: The next section contains some basic ideas about the Morris and Thorne wormhole and gives the definition of the evolving wormholes. Then we discuss the spatial geometry of the (2+1)-dimensional evolving wormhole. In Section three we discuss the dynamics of the evolving wormhole at the trapping horizon by introducing the phantom energy as a perfect fluid. Section four deals with the thermal interpretation of the field equations of (2+1)-dimensional evolving wormhole at the trapping horizon. Finally we present the conclusion in the last section.

## Preliminaries

### The Morris-Thorne Wormhole

Before going over the discussion of evolving wormhole, let us first recall some properties of static case. Morris and Thorne [5, 6] work out a general, static and spherically symmetric

traversable wormhole described by the metric

$$ds^2 = -e^{2\Phi(r)} dt^2 + \left[ \frac{dr^2}{1 - \frac{b(r)}{r}} + r^2(d\theta^2 + \sin^2 \theta d\phi^2) \right], \quad (1)$$

where  $\Phi(r)$  denotes the redshift function and  $b(r)$  is the shape function. The redshift function  $\Phi(r)$  must be finite throughout the spacetime in order to ensure the absence of horizons and singularities. The minimum radius  $r_0$  corresponds to the throat of the wormhole, where  $b(r_0) = r_0$  and the embedded surface is verticale. The proper radial distance is defined by

$$l(r) = \pm \int_{r_0}^r \frac{dr}{\sqrt{1 - b(r)/r}}, \quad (2)$$

and it must be finite throughout the wormhole spacetime, i. e.  $b(r)/r \leq 1$  for  $r \geq r_0$ . Here  $\pm$  signs represent the asymptotically flat regions which are connected by the wormhole's throat. The equality sign in  $b(r)/r \leq 1$  holds only at the throat. The both functions ( $\Phi(r)$  and  $b(r)$ ) must tend to a constant value as the radial coordinate goes to infinity. The shape function must satisfy the asymptotic flatness condition, i. e. as  $l \rightarrow \pm\infty$  (or equivalently,  $r \rightarrow \infty$ ) then  $b(r)/r \rightarrow 0$ . With these constraints we are able to see through an analysis of the embedding diagram of (1) in a Euclidean space, two asymptotically flat sections connected by a throat.

### Evolving Lorentzian Wormhole

In the cosmological set up, the evolving wormhole spacetime may be obtained by a simple generalization of the Morris and Thorne metric (1) to a time dependent metric given by [11, 13, 15, 33–35]

$$ds^2 = -e^{2\Phi(t,r)} dt^2 + a^2(t) \left[ \frac{dr^2}{1 - \frac{b(r)}{r}} + r^2(d\theta^2 + \sin^2 \theta d\phi^2) \right], \quad (3)$$

where  $a(t)$  is the scale factor of the universe. Worth noticing point is that the essential characteristics required for a wormhole geometry are still lies in the spacelike section. Notice that if  $b(r) \rightarrow 0$  and  $\Phi(t, r) \rightarrow 0$  the wormhole metric (3) becomes the flat Friedmann-Robertson-Walker (FRW) metric, and as  $a(t) \rightarrow \text{constant}$  it reduces to the static wormhole metric (1). For the construction of an evolving wormhole, one has to choose some ansatz for the redshift function  $\Phi(t, r)$ , the shape function  $b(r)$  or the scale factor  $a(t)$  and the others are

determined by employing some physical conditions. For instance the author [11] considered an exponential scale factor in order to explore the possibility that inflation might provide a natural mechanism for the enlargement of an initially small (possibly submicroscopic) wormhole to macroscopic size. Since we will be dealing with the dynamical spacetime in lower dimension, so by using the ansatz  $\Phi(t, r) = 0$  and  $b(r) = r_0^2/r$ , (where  $r_0$  is a finite radius of the wormhole's throat) as given by [5], the metric for (2+1)-dimensional evolving wormhole is given as

$$ds^2 = -dt^2 + a^2(t) \left[ \frac{dr^2}{1 - \frac{r_0^2}{r^2}} + r^2 d\theta^2 \right]. \quad (4)$$

### Spatial Geometry of the (2+1)-Dimensional Evolving Wormhole

To investigate the geometric nature of the (2+1)-dimensional wormhole given by the metric (4), we embed the spatial geometry of (2+1) wormhole into a flat 3-dimensional Euclidean space  $R^3$  [5] together with the study of expansion factor  $a(t)$  involved in our case. So using the spherical symmetry we can set an equatorial slice ( $\theta = \pi/2$ ) at some fix instant of constant time  $t = t_0$  which implies  $dt = 0$ . Hence the metric on the resulting two-surface is

$$ds^2 = \frac{a_0^2 dr^2}{1 - r_0^2/r^2} + a_0^2 r^2 d\phi^2, \quad (5)$$

where  $a(t_0) = a_0$  is the value of the scale factor at  $t = t_0$ . By substituting  $a_0 r = \bar{r}$  which implies  $a_0^2 dr^2 = d\bar{r}^2$ , the above metric (5) can be rewritten as

$$ds^2 = \frac{d\bar{r}^2}{1 - a_0^2 r_0^2 / \bar{r}^2} + \bar{r}^2 d\phi^2, \quad (6)$$

where we set  $a_0 r_0 = \bar{r}_0$  is the throat radius of the evolving wormhole at a specific time  $t = t_0$ . The throat radius  $\bar{r}_0 \geq r_0$  accordingly  $a_0 \geq 1$ . The 3-dimensional Euclidean space at  $t = t_0$  can be written in cylindrical coordinates  $(r, \phi, z)$  as

$$ds^2 = d\bar{z}^2 + d\bar{r}^2 + \bar{r}^2 d\phi^2,$$

or

$$ds^2 = (1 + d\bar{z}^2/d\bar{r}^2) d\bar{r}^2 + \bar{r}^2 d\phi^2. \quad (7)$$

In this study we use inflation to enlarge an initially small (possibly submicroscopic) wormhole. This inflated wormhole will have the same overall size and shape relative to the initial

$\bar{z}, \bar{r}, \phi$  as the initial wormhole had relative to the initial  $z, r, \phi$  embedding space coordinate system. This is just because we are considering a series of embedding spaces, each corresponds to a particular value of  $t = \text{constant}$  whose  $z, r$  coordinates scale with time. After comparing the metric (6) and (7), one can readily workout the equation of the embedded surface as

$$\frac{dz}{d\bar{r}} = \pm \frac{\bar{r}_0}{\sqrt{\bar{r}^2 - \bar{r}_0^2}}, \quad (8)$$

equivalently

$$\frac{dz}{dr} = \pm \frac{r_0}{\sqrt{r^2 - r_0^2}}. \quad (9)$$

The Eqs. (6) and (7) imply that the evolving wormhole will remain the same size in the  $\bar{z}, \bar{r}, \phi$  coordinates. So from equation (9) it is evident that at the throat radius  $r = r_0$ , the embedded surface is vertical for which  $\frac{dz}{dr} \rightarrow \infty$ . As  $r \rightarrow \infty$ ,  $\frac{dz}{dr} \rightarrow 0$  implies the embedded geometry is asymptotically flat. The equation (9) represents two embedded surfaces above and below the  $r$ -axis and these two surfaces join at  $r = r_0$ . To visualize the embedded diagram we integrate equation (9) which yields

$$z(r) = \pm r_0 \left[ \ln |r + \sqrt{r^2 - r_0^2}| - \ln |r_0| \right]. \quad (10)$$

In Figures (1-3), we have plotted the surface of revolution about the  $z$ -axis for a fixed radius  $r_0$  and different choices of the scale factor  $0 < a_0 < 1$ ,  $a_0 = 1$  and  $a_0 > 1$ , which correspond to respectively contracting, static and expanding wormholes.

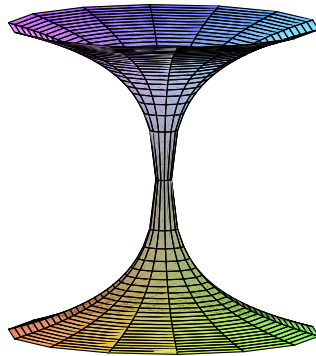


FIG. 1: The full visualization of the surface generated by the rotation of the embedded curve about the vertical axis. Chosen parameters are  $r_0 = 1$  and  $r = 1 \dots 10$

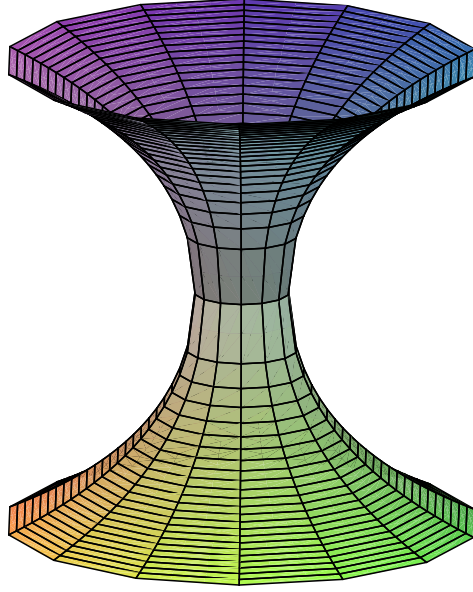


FIG. 2: The full visualization of the surface generated by the rotation of the embedded curve about the vertical axis. Chosen parameters are  $r_0 = 2$  and  $r = 2...10$

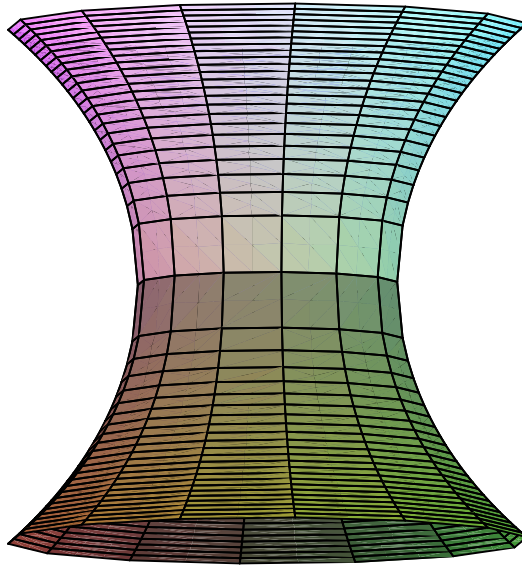


FIG. 3: The full visualization of the surface generated by the rotation of the embedded curve about the vertical axis. Chosen parameters are  $r_0 = 4$  and  $r = 4...10$



One can define the proper radial distance  $l(r) = \pm \int_{r_0}^r \frac{a(t)dr}{\sqrt{1-b(r)/r}}$ , which must be well behaved everywhere [5]. For our model, this parameter  $l$  at  $t = t_0$  for the upper part of the wormhole  $z > 0$  is

$$l(r) = a_0 \sqrt{r^2 - r_0^2}, \quad (11)$$

and for the lower part  $z < 0$

$$l(r) = -a_0 \sqrt{r^2 - r_0^2}. \quad (12)$$

In Figure 4, we have plotted the proper radial distance against  $r$  which shows that this function  $l(r)$  is a well-behaved quantity everywhere. It is clear from the diagrams that both, the size of the throat and the radial proper distance between the wormhole mouths increases with the passage of time. In our case, as mentioned earlier, far from the wormhole mouth the space is asymptotically flat. On the other hand, if we take the cosmological constant into account, then the resulting space could be de-Sitter or anti-de Sitter far from the mouth. In literature different form of scale factor are available, for instance, the authors [33] introduced a linear form of scale factor while in [11] different exponential forms of the scale factor are discussed and their cosmological aspects have also been studied. So the role of the scale factor becomes so important with clear indications that as the wormhole inflate, its throat size and proper length inflate along with the surrounding space.

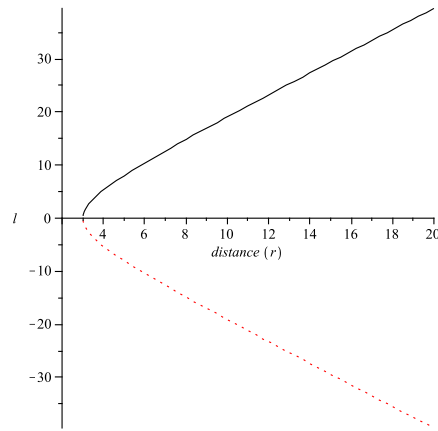


FIG. 4: The proper radial distance  $l$  is shown. The chosen parameters are  $a_0 = 2$  and  $r_0 = 3$ .

## Dynamics of the Evolving Wormhole at the Trapping Horizon

This section deals with some dynamical features of the (2+1)-dimensional evolving wormhole. Since the term event horizon is no longer possible for the wormhole spacetimes, but still we can introduce for them the term trapping horizon to discuss the dynamics of these peculiar geometries. It has been argued that the trapping horizon for a dynamical spacetime is a causal horizon associated with the gravitational entropy and the surface gravity [19, 36, 38]. The concept of trapping horizon for the wormhole geometries will lead us to explore some physical insights of these weird objects. So to start with, the spherically symmetric form of the metric (4) is

$$ds^2 = h_{ab}dx^a dx^b + \tilde{r}^2 d\theta^2, \quad (a, b = 0, 1) \quad (13)$$

where  $x^0 = t$ ,  $x^1 = r$ , and  $\tilde{r} = ar$  represents the radius of the sphere while the two dimensional metric is written as

$$h_{ab} = \text{diag}\left(-1, a(t)^2\left(1 - \frac{r_0^2}{r^2}\right)^{-1}\right). \quad (14)$$

In this section we revisit some of the terms and notations used in Hayward formalism based on the dual-null coordinates [20, 41]. The double Null coordinates by using metric (4) are constructed and are given by

$$d\xi^+ = -\frac{1}{\sqrt{2}}\left(dt - \frac{adr}{\sqrt{1 - r_0^2/r^2}}\right), \quad (15)$$

$$d\xi^- = -\frac{1}{\sqrt{2}}\left(dt + \frac{adr}{\sqrt{1 - r_0^2/r^2}}\right), \quad (16)$$

so with these coordinates, the wormhole metric (13) takes the form

$$ds^2 = -2d\xi^+ d\xi^- + \tilde{r}^2 d\theta^2. \quad (17)$$

It is easy to write

$$\partial_+ = \frac{\partial}{\partial \xi^+} = -\sqrt{2}\left(\frac{\partial}{\partial t} - \frac{\sqrt{1 - r_0^2/r^2}}{a} \frac{\partial}{\partial r}\right) \quad (18)$$

$$\partial_- = \frac{\partial}{\partial \xi^-} = -\sqrt{2}\left(\frac{\partial}{\partial t} + \frac{\sqrt{1 - r_0^2/r^2}}{a} \frac{\partial}{\partial r}\right), \quad (19)$$

where the minus signs ensure that  $\partial_{\pm}$  are future pointing. One can define the expansions as

$$\theta_{\pm} = \frac{2}{\tilde{r}} \partial_{\pm} \tilde{r}. \quad (20)$$

Since the sign of  $\theta_+\theta_-$  is invariant, one can say that a sphere is trapped if  $\theta_+\theta_- > 0$ , i. e.

$$(H^2\tilde{r}^4 - \tilde{r}^2 - a^2r_0^2) > 0, \quad (21)$$

untrapped if  $\theta_+\theta_- < 0$  i. e.

$$(H^2\tilde{r}^4 - \tilde{r}^2 - a^2r_0^2) < 0, \quad (22)$$

and marginal if  $\theta_+\theta_- = 0$  i. e.

$$(H^2\tilde{r}^4 - \tilde{r}^2 - a^2r_0^2) = 0. \quad (23)$$

If the expansion  $\theta_+ > 0$  and  $\theta_- < 0$  is locally fixed on an untrapped sphere, then  $\partial_+$  and  $\partial_-$  are also fixed as the outgoing and ingoing null normal vectors (or the contrary if the orientation  $\theta_+ < 0$  and  $\theta_- > 0$  is considered). A marginal sphere with  $\theta_+ = 0$ , i. e.

$$-\frac{2\sqrt{2}}{\tilde{r}^2} \left( H\tilde{r} - \sqrt{1 - a^2r_0^2/\tilde{r}^2} \right) = 0, \quad (24)$$

is future if  $\theta_- < 0$ , i. e.

$$-\frac{2\sqrt{2}}{\tilde{r}^2} \left( H\tilde{r} + \sqrt{1 - a^2r_0^2/\tilde{r}^2} \right) < 0, \quad (25)$$

past if  $\theta_- > 0$ , i. e.

$$-\frac{2\sqrt{2}}{\tilde{r}^2} \left( H\tilde{r} + \sqrt{1 - a^2r_0^2/\tilde{r}^2} \right) > 0 \quad (26)$$

and bifurcating if  $\theta_- = 0$ , i. e.

$$-\frac{2\sqrt{2}}{\tilde{r}^2} \left( H\tilde{r} + \sqrt{1 - a^2r_0^2/\tilde{r}^2} \right) = 0. \quad (27)$$

This marginal sphere is outer if  $\partial_-\theta_+ < 0$  (equivalently  $\partial_-\partial_+\tilde{r} < 0$ ) i. e.

$$-\frac{2\sqrt{2}}{\tilde{r}^2} \left( \dot{H} + 2H - \frac{a^2r_0^2}{\tilde{r}^4} \right) < 0, \quad (28)$$

inner if  $\partial_-\theta_+ > 0$  (equivalently  $\partial_-\partial_+\tilde{r} > 0$ ) i. e.

$$-\frac{2\sqrt{2}}{\tilde{r}^2} \left( \dot{H} + 2H - \frac{a^2r_0^2}{\tilde{r}^4} \right) > 0 \quad (29)$$

and degenerate if  $\partial_-\theta_+ = 0$  (equivalently  $\partial_-\partial_+\tilde{r} = 0$ ) i. e.  $\dot{H} + 2H - \frac{a^2r_0^2}{\tilde{r}^4} = 0$ . A hypersurface foliated by marginal sphere is called a trapping horizon and has the same

classification as the marginal sphereres. Hence the expression for the trapping horizon yields

$$\theta_+ = \left( H\tilde{r} - \sqrt{1 - a^2 r_0^2 / \tilde{r}^2} \right) = 0. \quad (30)$$

Let us now consider the Einstein field equations

$$G_{mn} = -\pi T_{mn}, \quad (m, n = 0, 1, 2) \quad (31)$$

where  $G_{mn}$  is the Einstein tensor and  $T_{mn}$  is the energy-momentum tensor of the matter fields. We have used  $G = 1/8$  in Eq. (31). One motivating feature of discussion of the evolving wormhole is the possibility of sustaining of these geometries by means of exotic matter made out of phantom energy. The later is used as a main source to explain the late time accelerated expansion of the universe [37]. Since this energy violates the NEC, so it is used as a most appropriate ingredient that could sustain the wormhole geometries. So we take phantom energy as a perfect fluid for the evolving wormhole given by

$$T^{mn} = (\rho + p)u^m u^n + p g^{mn}, \quad (32)$$

where  $\rho(t)$  and  $p(t)$  are time dependent energy density and pressure while  $u^m = (1, 0, 0)$  is the comoving three velocity of the fluid. The energy conservation condition  $T_{;m}^{mn} = 0$ , yields  $\dot{\rho} + 2H(\rho + p) = 0$ . Solving the Einstein field equation (31), in the background of wormhole geometry (4), one can get by utilizing  $G_0^0$  and  $G_1^1$ , the Friedman-like field equation

$$\dot{H} + \frac{r_0^2}{a^2 r^4} = -\pi(\rho + p), \quad (33)$$

where  $H = \dot{a}/a$ , is the Hubble parameter and overdot indicates the derivative with respect to the cosmic time. The explicit form of the trapping horizon can be evaluated by using the relation  $\theta_+ = 0$ , which after simplification yields

$$H^2 \tilde{r}_A^4 - \tilde{r}_A^2 + a^2 r_0^2 = 0, \quad \text{where } \tilde{r}_A = a(t)r \quad (34)$$

which is quadratic in  $\tilde{r}_A^2$ . It can be seen from Eq. (34) that when  $r_0 = 0$  we have namely a flat FRW universe, and the wormhole trapping horizon  $\tilde{r}_A$  takes the same value as the Hubble horizon,  $\tilde{r}_A = 1/H$ . The Hubble parameter in terms of the wormhole trapping radius is  $H^2 = (1/\tilde{r}_A^2 - a^2 r_0^2 / \tilde{r}_A^4)$ , and its time derivative yields

$$\dot{H} = -\frac{\dot{\tilde{r}}_A}{H \tilde{r}_A^3} \left( 1 - \frac{2a^2 r_0^2}{\tilde{r}_A^2} \right) - \frac{a^2 r_0^2}{\tilde{r}_A^4}. \quad (35)$$

The trapping horizons of the wormhole metric (13) are described by the roots of the Eq. (34) which yields

$$\begin{aligned}\tilde{r}_{A+}^2 &= \frac{1 + \sqrt{1 - 4H^2 a^2 r_0^2}}{2H^2}, \\ \tilde{r}_{A-}^2 &= \frac{1 - \sqrt{1 - 4H^2 a^2 r_0^2}}{2H^2}.\end{aligned}\tag{36}$$

There are three cases depending upon the roots; (a) Two distinct real roots ( $1 - 4H^2 a^2 r_0^2 > 0$ ) refer as a usual wormhole geometry, (b) two repeated real roots ( $1 - 4H^2 a^2 r_0^2 = 0$ ) called as the ‘extreme wormhole’ geometry, (c) no real roots ( $1 - 4H^2 a^2 r_0^2 < 0$ ) imply the ‘naked wormhole’. The notion of nakedness refers to the absence of dynamical trapping horizon. If we assume that  $0 < r_0^2 \ll 1$ , and neglecting  $O(r_0^4)$ , it is possible to simplify the expressions for  $\tilde{r}_{A+}$  and  $\tilde{r}_{A-}$ , which give

$$\tilde{r}_{A+}^2 = \frac{1}{H^2} - a^2 r_0^2, \quad \tilde{r}_{A-}^2 = a^2 r_0^2.\tag{37}$$

It is evident from equation (37) that the outer trapping horizon will contract while the inner horizon will expand. It is also interesting to note that the sum and product of squares of wormhole horizons satisfy  $\tilde{r}_{A+}^2 + \tilde{r}_{A-}^2 = \frac{1}{H^2}$ , and  $\tilde{r}_{A+}^2 \tilde{r}_{A-}^2 = \frac{a^2 r_0^2}{H^2}$ , respectively. From Eq. (36) the trapping horizon  $\tilde{r}_{A+}$  and  $\tilde{r}_{A-}$  coincide at the extreme case leading to the extreme trapping horizon  $\tilde{r}_A = 1/\sqrt{2}H$ . Moreover in this case, the wormhole parameters leading to  $\dot{a}^2 = \frac{1}{4r_0^2}$ , which upon integration gives  $a(t) = \pm \frac{t}{2r_0}$ . Here the constant of integration is taken to be zero. It shows that the wormhole is expanding uniformly if  $a(t) > 0$  and contracting if  $a(t) < 0$ . Also the naked wormhole is obtained if the discriminant  $1 - 4H^2 a^2 r_0^2 < 0$ , which yields  $\dot{a} > \frac{1}{2r_0}$ .

### (2+1)-Wormhole Thermodynamics

In this section, we discuss the horizon thermodynamics of (2+1) evolving wormhole. Hayward first introduced a formalism for defining thermal properties of black holes in terms of measurable quantities. This formalism also works for the dynamical black holes which consistently recover the results obtained by global considerations using the event horizon as in the static case. A fascinating and rather surprising feature emerges if one recognize that the static wormhole (where it is not possible to infer any property similar to those found in black holes physics using global considerations) reveals thermodynamic properties

analogous to the black holes if one considers the local quantities. It is also important to note that the non-vanishing surface gravity at the wormhole throat characterized by a non-zero temperature for which one would expect that wormhole should emit some sort of thermal radiation. Since in our case, the wormhole is defined to be traversable therefore any matter or radiation could pass through it from one spacetime to an other (or from a region to another of the same spacetime) which finally come out in the latter spacetime. One can discriminate the phenomenon in which a radiation travel through the wormhole which follows a path allowed by classical general relativity whereas , thermal radiation from the horizon is essentially quantum mechanical process. Therefore, in case of no matter or radiation would pass through the wormhole throat classically, the existence of a trapping horizon of the evolving wormhole would produce a quantum thermal radiation. We assume that the entropy associated with the outer trapping horizon  $\tilde{r}_{A+}$ , is proportional to the horizon area analogous to the black hole entropies. So in this case the entropy of the wormhole becomes

$$S = 4\pi\tilde{r}_{A+}. \quad (38)$$

The surface gravity is defined as [38]

$$\kappa = \frac{1}{2\sqrt{-h}}\partial_a(\sqrt{-h}h^{ab}\partial_b\tilde{r}), \quad (39)$$

where  $h$  is the determinant of metric  $h_{ab}$  (13). The direct calculation of the surface gravity from Eq. (39) at the wormhole horizon  $\tilde{r}_{A+}$  yields

$$\begin{aligned} \kappa &= -\frac{\tilde{r}_{A+}}{2}\left(\dot{H} + 2H^2 - \frac{a^2r_0^2}{\tilde{r}_{A+}^4}\right), \\ &= -\frac{1}{\tilde{r}_{A+}}\left(1 - \frac{\dot{\tilde{r}}_{A+}}{2H\tilde{r}_{A+}}\right)\left(1 - \frac{2a^2r_0^2}{\tilde{r}_{A+}^2}\right). \end{aligned} \quad (40)$$

The factor  $-\frac{1}{\tilde{r}_{A+}}\left(1 - \frac{\dot{\tilde{r}}_{A+}}{2H\tilde{r}_{A+}}\right)$ , in (41) is the general expression for the surface gravity of FRW universe while the second factor  $\left(1 - \frac{2a^2r_0^2}{\tilde{r}_{A+}^2}\right)$  appears due to the wormhole geometry. It is to be noted that when  $r_0$  approaches to zero, the expression for the surface gravity reduces to the expression for FRW universe. It is important to mention that the surface gravity vanishes at extreme case which is consistent with the extreme black hole case.

Now we can compute the horizon temperature  $T$  for the evolving wormhole with the help of the relation

$$T = -k/2\pi. \quad (41)$$

It is evident from Eq. (41) that the horizon temperature may negative, however, one can avoid any possible physical implication of it by claiming that it would be a problem only if one is at the trapping horizon. In addition, one must consider that the radiation infalling in one of the wormhole mouths will travel classical trajectory to re-appear at the other mouth of the wormhole as an outgoing radiation. Similarly, the same process would also take place at the other end. So finally we see an outgoing radiation which surely be unavoidable with negative temperature in the universe whenever a wormhole is investigated. Moreover, as mentioned earlier that the phantom energy would be a source to construct the traversable wormhole and it has been argued that the phantom energy may be characterized by a negative temperature [39]. Therefore, the above result (41) indicates that wormholes sustained by phantom energy should emit thermal radiations with negative temperature analogous to the black holes constructed by the ordinary matter with positive temperature.

After substituting the value of surface gravity (40) in Eq. (41), we obtain

$$T = \frac{1}{2\pi\tilde{r}_{A+}} \left(1 - \frac{\dot{\tilde{r}}_{A+}}{2H\tilde{r}_{A+}}\right) \left(1 - \frac{2a^2r_0^2}{\tilde{r}_{A+}^2}\right). \quad (42)$$

We have determined the expression for the temperature of the evolving wormhole under consideration. Now our next step is to rewrite the Friedman-like equation (33) as a first law of thermodynamics. So for this let us first we write down the Friedman-like equation (33) at the trapping horizon  $\tilde{r}_{A+}$ . To do so, we put the value of  $\dot{H}$  from Eq. (35) in (33), to get

$$\left(1 - \frac{2a^2r_0^2}{\tilde{r}_{A+}^2}\right) d\tilde{r}_{A+} = \pi H \tilde{r}_{A+}^3 (\rho + p) dt. \quad (43)$$

On multiplying both sides of the above equation by a factor  $\left(1 - \frac{\dot{\tilde{r}}_{A+}}{2H\tilde{r}_{A+}}\right)$  and arranging the terms, we get

$$\frac{1}{2\pi\tilde{r}_{A+}} \left(1 - \frac{\dot{\tilde{r}}_{A+}}{2H\tilde{r}_{A+}}\right) \left(1 - \frac{2a^2r_0^2}{\tilde{r}_{A+}^2}\right) d(4\pi\tilde{r}_{A+}) = 2\pi H \tilde{r}_{A+}^2 \left(1 - \frac{\dot{\tilde{r}}_{A+}}{2H\tilde{r}_{A+}}\right) (\rho + p) dt. \quad (44)$$

From equation (44), one can recognize that the term on the left hand side is  $TdS$ , where  $S = 4\pi\tilde{r}_{A+}$ , is the entropy of the wormhole. So the above equation reduces to

$$TdS = 2\pi H \tilde{r}_{A+}^2 \left(1 - \frac{\dot{\tilde{r}}_{A+}}{2H\tilde{r}_{A+}}\right) (\rho + p) dt. \quad (45)$$

Now we consider the total matter-energy  $E = \rho A$ , surrounded by the trapping horizon  $\tilde{r}_{A+}$  of the evolving wormhole. Taking the differential of  $E$  and using the energy conservation

relation, we get

$$dE = 2\pi\tilde{r}_{A+}\rho d\tilde{r}_{A+} - 2\pi\tilde{r}_{A+}^2 H(\rho + p)dt. \quad (46)$$

Using Eqs. (45) and (46), we finally achieve

$$dE = -TdS + WdA, \quad (47)$$

where  $W = (\rho - p)/2$  is the work density which is defined by  $W = -\frac{1}{2}T^{ab}h_{ab}$  [16–19, 40]. The work term  $WdA$  can be interpreted as the work done against the pressure at the trapping horizon. The expression (47) is recognized as the unified first law of thermodynamics [16, 19, 41]. The negative sign appears in the first term of right hand side of Eq. (47) is mainly appears due to the exotic matter which gets energy from the spacetime itself. Hence one can interpret that the change in the internal energy  $dE$  of the evolving wormhole equals the sum of the energy removed from the system (wormhole) plus the energy consumed in performing work done against the pressure. In short, by employing the entropy proportional to the horizon area together with the total matter energy density within the trapping horizon of the wormhole, we are able to show that the Friedman-like equation (33) can be expressed as a thermodynamic identity. So the notions of temperature and entropy can be associated with the trapping horizon of the wormhole analogous to the apparent horizon of FRW universe.

## Conclusion

It has been shown that the field equations in various theory of gravities can be recast as a first law of thermodynamics at the horizons of a class of spacetime geometries. An enormous work dealing with the horizon thermodynamics is available [38, 41]. On the other, Hayward has shown that these geometries (black holes) can be investigated for the thermodynamic characteristics by taking the trapping horizons into account. In the Hayward formalism, the black holes and wormhole spacetimes can be treated on equal footing in order to study their dynamical properties on the basis of trapping horizons. So our aim has been to visualize the thermodynamic properties of the evolving wormholes at the trapping horizons. In this regard, it is shown that the field equations of (2+1)-dimensional evolving wormhole can be expressed as a first law of thermodynamics  $dE = -TdS + WdA$ , at the trapping horizon. Here  $E = \rho A$  is the total energy of the matter inside the horizon. Here  $W = (\rho - p)/2$  and  $A = \pi\tilde{r}_A^2$  are the work density and area respectively. We also discussed the embedded



diagram to study the evolutionary behavior of the evolving wormhole.

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